Glimpses on functionals with general growth

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Outline

Our model case is:

$$F(u) = \int_\Omega f(Du)dx$$

- $\Omega \subset \mathbb{R}^n$ bounded open set, $u : \Omega \to \mathbb{R}^N$
- f satisfies some growth condition:

$$|z|^p \leq f(z) \leq c(1 + |z|)^p$$

- f is convex and C^2.

A function $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ (Sobolev space) is a local minimizer for F if

$$F(u, spt\eta) \leq F(u + \eta, spt\eta) \quad \forall \eta \in C^1_0(\Omega, \mathbb{R}^N).$$

Framework: David Hilbert's 19th problem

Does every lagrangian partial differential equation of a regular variational problem have the property of exclusively admitting analytic integrals?

Whether "regular" variational problems admit only analytic solutions?
F is called a regular variational integral if $f \in C^2(\mathbb{R}^{n\times N})$ and

$$\nu|B|^2 \leq D^2 f(A)[B,B] \leq L|B|^2 \quad 0 < \nu \leq L$$

Theorem

($W^{2,2}_{loc}$-regularity) Any minimizer $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is in fact $W^{2,2}_{loc}$.

In the special case that f is quadratic a bootstrap argument gives $u \in C^\infty$!
Campanato shows $u \in W^{2,2+\delta}$, with $\delta = \delta(n, \frac{L}{\nu})$
Kristensen and Melcher proved a dimension free value for δ.

De Giorgi -Nash-Moser theorem

Consider a uniformly elliptic equation (N=1) with measurable coefficients:

$$-div(a(x)\nabla u) = 0$$

then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, \frac{L}{\nu})$.
Nash proved the result for parabolic equations.
Moser proved Harnack's inequality.

$$\lim_{\frac{L}{\nu}\to\infty} \alpha = 0$$

Theorem

Let $u \in W^{1,p}(\Omega)$ be a weak solution of the equation $\operatorname{div} a(x, u, Du) = 0$ under the monotonicity and growth conditions of p-growth:

$$|a(x, v, z)| \leq L(1 + |z|^{p-1}),\ \nu|z|^p - L \leq \langle a(x, v, z), z\rangle$$

then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, p, \frac{L}{\nu})$

For functionals: Without any differentiability assumption, the result holds true
Frehse
Giaquinta Giusti (hole filling technique of Widman)
Di Benedetto Trudinger proved Harnack for functions in De Giorgi classes.

$C^{1,\alpha}$-regularity

- Giaquinta, Giusti;
- Ivert;
- Manfredi;
- Lewis;
- Di Benedetto;
- Tolksdorff.

The vectorial case

A well known result of Uraltseva and K. Uhlenbeck (77) states that the $C^{1,\alpha}$-regularity holds for local minimizers if the integrand function is of the type

$$f(|z|)$$

for a convex function f of p-growth, with $p \geq 2$.

Known results

- Uraltseva, Uhlenbeck '77
- Giaquinta-Modica '86
- Acerbi-Fusco '89 $1 < p < 2$

What happens if the power function t^p is replaced by a general convex function $\phi(t)$? (general growth)

Orlicz-Sobolev

- L^ϕ : $f \in L^\phi$ iff there exists $K > 0$ such that $\int \phi(\frac{|f|}{K})\, dx < \infty$
- $W^{1,\phi}$: $f \in W^{1,\phi}$ iff $f, Df \in L^\phi$.

Known results

- Marcellini '89-'96 general growth
- Lieberman : scalar case '91 ; vectorial case '93
- Mingione-Siepe '99
- Esposito-Mingione '00 nearly linear growth
- Fuchs-Mingione '00
- Marcellini-Papi '06
- Bildhauer Fuchs

Marcellini's,Marcellini-Papi's approach: Euler system

$$u \in W^{1,\infty}, A \in C^1 \Longrightarrow u \in C^{1,\alpha}$$

without excess decay estimate!

Excess functional:

$$\Phi(x_0, r) = \fint_{B_r} |V(Du) - V(Du)_{x_o,r}|^2 \, dx$$

where $V(z) = |z|^{\frac{p-2}{2}} z$.

Question

What are suitable assumptions on ϕ that guarantee everywhere $C^{1,\alpha}$-regularity for local minimizers?

ϕ *N*-function

- $\phi(0) = 0$
- ϕ' right continuous, non-decreasing
- $\phi'(0) = 0$, $\phi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \phi'(t) = \infty$.

- $\phi \in C^1([0,\infty)) \bigcap C^2((0,\infty))$
- H1. $\phi'(t) \sim t\phi''(t)$ uniformly in $t > 0$
- H2. Hölder continuity for ϕ''

$$|\phi''(s+t) - \phi''(t)| \le c\,\phi''(t)\left(\frac{|s|}{t}\right)^{\beta} \quad \beta > 0$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

- H1. $\Longrightarrow \Delta_2(\phi, \phi^*) < \infty$ (ϕ^* is the conjugate)

Δ_2 condition

$$\phi \in \Delta_2 \Leftrightarrow \exists c_1 > 0 : \phi(2t) \leq c_1 \phi(t)$$

Examples

- $\phi(t) = t^p \quad \forall p > 1$
- $\phi(t) = t^p \log^\alpha(e + t)$
- $\phi(t) = t^p \log\log(e + t)$

Main Theorem

Let $u \in W^{1,\phi}_{loc}(\Omega, \mathbb{R}^n)$ local minimizer for

$$\int_\Omega \phi(|Du|)dx$$

with ϕ like before

$$\Downarrow$$

"excess decay estimate"

$$\fint_{B_\rho} |V(Du) - (V(Du))_\rho|^2 \leq c(\frac{\rho}{R})^\alpha \fint_{B_R} |V(Du) - (V(Du))_R|^2 \, \forall \rho < R$$

$$\Downarrow$$

Du locally Hölder continuous

Nonlinear quantities

$$A(Q) = \phi'(|Q|)\frac{Q}{|Q|} \quad V(Q) = \psi'(|Q|)\frac{Q}{|Q|} \quad \psi'(t) = \sqrt{\phi'(t)t}$$

The nonlinearity of the problem is inserted in V !

$$A(Q) \cdot Q \sim |V(Q)|^2 \sim \phi(|Q|)$$

$$(A(P) - A(Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2$$

uniformly in $P, Q \in \mathbb{R}^{N \times n}$.

For $\lambda > 0$ shifted function :

$$\phi'_\lambda(t) = \frac{\phi'(\lambda + t)t}{\lambda + t}$$

ϕ_λ inherits all properties of ϕ uniformly in λ

Main steps

- 1. Poincaré and Caccioppoli in the Orlicz-Sobolev setting;

$$\Downarrow$$

Gehring-type result

- 2. Bernstein-Uhlenbeck trick: $\phi(|Du|)$ is a subsolution of a uniformly elliptic equation

$$\Downarrow$$

weak Harnack inequality

- 3. Excess decay estimate

$$\Downarrow$$

conclusion using the integral characterization of Campanato spaces.

Reverse Hölder:$\exists\, q_1 > 1$ such that $\forall q \in [1, q_1]$

$$\left(\fint_B |V(Du)|^{2q} dx\right)^{\frac{1}{2q}} \leq c\left(\fint_{2B} |V(Du)|^2 dx\right)^{\frac{1}{2}}$$

and

Reverse Hölder for the oscillation:

$$\fint_B |V(Du) - V(Q)|^2 dx \leq c\left(\fint_{2B} |V(Du) - V(Q)|^{2\theta} dx\right)^{\frac{1}{2\theta}}$$

Using difference quotient technique: $V(Du) \in W^{1,2}$ and

$$\fint_B |DV(Du)|^2 dx \leq \frac{c}{R^2} \fint_{2B} |V(Du)|^2 dx$$

Uhlenbeck trick

We use the approximated functionals $F_\lambda = \int_\Omega \phi_\lambda(|Dv|)dx$

$$\Downarrow$$

$\phi_\lambda(|Du_\lambda|)$ subsolution of a problem whose coefficients satisfy

$$c_0|\xi|^2 \leq \sum_{kl} G_\lambda^{kl}(Q)\xi_k\xi_l \leq c_1|\xi|^2$$

$$\Downarrow$$

$$\sup_B \phi_\lambda(|Du_\lambda|) \leq c \fint_{2B} \phi_\lambda(|Du_\lambda|) (DeGiorgi - Nash - Moser)$$

$$\Downarrow$$

$$\sup_B \phi(|Du|) \leq c \fint_{2B} \phi(|Du|) \quad \left((\lambda, Q) \to V_\lambda^{-1}(Q) \text{ is continuous}\right)$$

$$\Downarrow$$

$$\fint_{\frac{1}{2}B} |V(Du) - \langle V(Du)\rangle_{\frac{1}{2}B}|^2 \leq c \left(\sup_B \phi(|Du|) - \sup_{\frac{1}{2}B} \phi(|Du|) \right)$$

Excess decay estimate

Using the Hölder continuity of ϕ'' we get:
$\forall \tau \in (0,1) \exists \varepsilon_0(\tau) \in (0,1)$ such that

$$\Phi(u,R) \leq \varepsilon_0 \sup_{B_{R/2}} \phi(|Du|) \Longrightarrow \Phi(u,\tau R) \leq c\tau^2 \Phi(u,R)$$

$$\Phi(u,R) = \fint_{B_R} |V(Du) - (V(Du))_R|^2 dx$$

How should we remove the “smallness " assumption?
We prove an alternative using the weak Harnack inequality.

- Using a standard iteration tecnique, we prove that $\exists \alpha > 0 : \forall B \subset \Omega \;\; \Phi(u,\rho) \leq c(\frac{\rho}{R})^\alpha \Phi(u,R) \quad \forall \rho < R$
- From Campanato characterization of Hölder continuous functions, we get $V(Du)$ locally Hölder continuous;
- Using that V^{-1} Hölder continuous, we conclude: Du locally Hölder continuous.

Remark (D.Breit, A.Verde, B.S. 2011)

If φ satisfies

- Δ_2-condition;
- $\widehat{\varepsilon}\frac{\varphi'(t)}{t} \leq \varphi''(t) \leq a(1+t^2)^{\frac{\omega}{2}}\frac{\varphi'(t)}{t}, \quad \omega > 0$
- the Hölder continuity of $\varphi''(t)$,

then there exists $\sigma > 0$ such that $u \in C^{1,\sigma}(\Omega;\mathbb{R}^N)$.

Example

We can consider convex functions that oscillate like the following example in Marcellini-Papi :

$$\varphi(t) = \begin{cases} t^p & , \quad t \leq \tau_0, \\ t^{\frac{p+q}{2}+\frac{q-p}{2}\sin\log\log\log t} & , \quad t > \tau_0; \end{cases}$$

where τ_0 is such that $\sin\log\log\log\tau_0 = -1$.

What happens if we remove the “radial structure"?

Hyp.: f smooth uniformly convex function with uniformly bounded second derivatives

Classical results: smooth minimizers, (Morrey, De Giorgi, Nash)

- $n = 2, N \geq 1$ or
- $n \geq 2, N = 1$

Counterexamples (Necas ’77)

$n > 2, N > 1 \Longrightarrow$ non smooth minimizers, but Lipschitz continuous .

Recent results, Sverak-Yan (2002)

Using “null Lagrangian" approach, they construct counterexamples showing that there exist regular variational integrals such that the minimizers must be:

- non-Lipschitz if $n \geq 3$, $N \geq 5$;
- unbounded if $n \geq 5$, $N \geq 14$.

Mooney, Savin 2015

Example of singular minimizer for $n = 3$ and $m = 2$.

- Asymptotically convex problems, (Chipot Evans):
- Elliptic sytems with ϕ-growth
- Quasiconvex problems

Given $H(\xi) = (1+|\xi|^2)^{\frac{p}{2}}$ we say that f is C^2 asymptotically convex if

$$\forall \varepsilon > 0, \exists \gamma_\varepsilon > 0 : |\frac{\partial^2 f}{\partial \xi^2}(\xi) - \frac{\partial^2 H}{\partial \xi^2}(\xi)| \leq \varepsilon |\xi|^{p-2}$$

whenever $|\xi| > \gamma_\varepsilon$.

Question

Which kind of regularity can we expect for local minimizers of

$$F(u) = \int_\Omega f(Du)dx?$$

Local Lipschitz regularity

- Chipot-Evans '86 $p = 2$
- Giaquinta-Modica '86 $p \geq 2$
- Leone-Passarelli di Napoli-V. '07 $1 < p < 2$
- Raymond '91, Kristensen-Taheri '03, Dolzman-Kristensen '05,
- Dolzman-Kristensen-Zhang,
- Scheven-Schmidt '09
- Carozza-Passarelli-Schmidt-Verde '10

Main Theorem

$$F(u) = \int_\Omega f(Du)dx$$

- $f \in C^2(\mathbb{R}^{nN})$
- $|D^2 f(\xi)| \leq c\phi''(|\xi|), \forall \xi \in \mathbb{R}^{nN} \setminus \{0\}$
- $\lim_{|\xi|\to\infty} \frac{|D^2 f(\xi) - D^2\phi(|\xi|)|}{\phi''(|\xi|)} = 0$

If $u \in W^{1,\phi}_{loc}(\Omega, \mathbb{R}^N)$ local minimizer for F, then Du is locally bounded.

$$\sup_B \phi(|Du|) \leq c(1 + \fint_{2B} \phi(|Du|)dx)$$

The proof is achieved comparing our minimizer with the minimizer of the model functional for which we have the excess decay estimate!

What about C^1 asymptotically convex problems?
Counterexample of Dolzmann, Kristensen, Zhang:$n = N = 2, p = 2$

$$\int_B dist^2(\nabla u, SO(2)),$$

$B \subset \mathbb{R}^2$ unit disc. The quasiconvex envelope F is C^1 asymptotically convex; $F'(\xi) = 2(\xi - c(\xi))$, with c bounded Lipschitz. There exists a minimizer in $W_0^{1,2}(B, \mathbb{R}^2)$ with unbounded gradient near 0 (also minimizer for F): it is $\frac{1}{4}\bar{z}\log|z|^2 \in W^{1,BMO}$!

Scheven-Schmidt result:
f locally bounded Borel integrand + asymptotically regular

$$\Downarrow$$

Ω can be decomposed into three disjoint sets:

- H is open u is $C^{1,\alpha}, \forall\alpha$;
- B_L $\quad x \in B_L$ is a Lebesgue point and $|Du| \leq L$;
- Σ is negligible set.

H and the interior of B_L are contained in the regular set, that is dense in Ω. But $S(u) \bigcap \partial B_L$ can have positive measure!
It is a result of partial Lipschitz regularity.
For $N = 1$ or $n = 2$ they show everywhere regularity.

Elliptic systems of ϕ-growth

What about Elliptic systems of ϕ-growth? Consider a system:

$$divA(x, \nabla u) = 0$$

for a vector field $A\colon \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ such that:

- A is Lipschitz continuous with respect to P
  $|A(x,P) - A(x,Q)| \leq L\,\phi''(\mu + |P| + |Q|)\,|P - Q|$
- $\nabla_{N\times n}A$ is Hölder continuous with some exponent α for $|Q| < \frac{1}{2}|P|$
  $|\nabla_{N\times n}A(x, P+Q) - \nabla_{N\times n}A(x,P)| \leq L\phi''(|P|)\left(|\frac{Q}{P}|\right)^{\alpha}$
- A is degenerate monotone:
  $\langle A(x,P) - A(x,Q), P - Q\rangle \geq \nu\,\phi''(\mu + |P| + |Q|)\,|P - Q|^2$
- A is Hölder continuous with respect to its first argument with exponent $\beta \in (0,1)$.

Question

Which kind of regularity we can expect for solutions of systems of ϕ-growth

$$divA(x,\nabla u) = 0?$$

Partial regularity

Hölder continuity of the gradient in a set whose complement has Lebesgue measure zero.(Hausdorff dimension)

Non degenerate case

Partial regularity: The basic idea is to linearize the problem near the gradient average.
We have different methods to implement a local linearization scheme:

- Indirect method via blow up techniques: Morrey,Giusti-Miranda, Evans, Acerbi-Fusco, Hutchinson,Hamburger..
- A-harmonic approximation method: De Giorgi (minimal surfaces),Simon, Duzaar-Steffen (geometric measure theory), Duzaar-Mingione,Duzaar-Gastel-Grotowski, Duzaar-Grotowski-Kronz,...

Degenerate case

Partial regularity: When we linearize near the gradient average, it may happen that $(Du)_{x,r}$ is near the origin or even 0 so that the linearized problem loses the ellipticity!
Idea for partial regularity:

- when Du is far from 0, then one can linearize as before;
- when Du is near 0, then one directly compares u with minimizers of the model case functional $\int_\Omega |Dv|^p dx$ via "p-harmonic approximation" (see notes of Lecture 3&4.)

A function $u \in W^{1,2}(\Omega)$ is weakly harmonic on Ω iff

$$\int_\Omega \nabla u \nabla \phi = 0 \, \forall \phi \in C_o^\infty(\Omega)$$

Weyl's Theorem If u is weakly harmonic, then the L^2 class of u has a representative which is harmonic.

Harmonic Approximation Lemma

Let B a ball in $\mathbb{R}^n$. For each $\varepsilon > 0$ there is $\delta = \delta(n, \varepsilon)$ such that if $u \in W^{1,2}(B)$, $\int_B |\nabla u|^2 \leq 1$ and

$$|\int_B \nabla u \nabla \phi| \leq \delta \sup |\nabla \phi|, \forall \phi \in C_o^\infty(B)$$

then there is a harmonic function h on B such that $\int_B |\nabla h|^2 \leq 1$ and

$$\int_B |h - u|^2 \leq \varepsilon$$

In this context of more general growth we prove a ϕ-harmonic approximation that also in case of powers give a new approximation in terms of the gradients

p-harmonic approximation

For every $\varepsilon > 0$ and $\theta \in (0,1)$, $\exists \delta = \delta(\varepsilon, \theta, \phi) > 0$ s.t. if $u \in W^{1,p}(B, \mathbb{R}^N)$ is *almost p-harmonic* i.e. $\forall \xi \in C_0^\infty(B, \mathbb{R}^N)$

$$\left| \fint_B |\nabla u|^{p-2} \langle \nabla u, \nabla \xi \rangle \, dx \right| \leq \delta \left(\fint_B |\nabla u|^p \, dx + \|\nabla \xi\|_\infty^p \right),$$

then the unique p-harmonic map h with $h = u$ on ∂B satisfies

$$\left(\fint_B |V(\nabla u) - V(\nabla h)|^{2\theta} \, dx \right)^{\frac{1}{\theta}} < \varepsilon \fint_B |\nabla u|^p \, dx.$$

The proof is based on a generalization of the Lipschitz approximation Lemma in the context of Orlicz spaces.

Lipschitz approximation Lemma

Ω bounded domain and $w \in W_0^{1,p}(\Omega, \mathbb{R}^N)$. For every $m_0 \in \mathbb{N}$ and $\gamma > 0$ there exists $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $w_\lambda \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ satisfies

- $\|w_\lambda\|_\infty \leq c\,\lambda$
- $\int_\Omega |\nabla w_\lambda|^p \chi_{\{w_\lambda \neq w\}}\, dx \leq c \int_\Omega \lambda^p \chi_{\{w_\lambda \neq w\}})\, dx$
- $\leq \frac{c}{m_0} \int_\Omega |\nabla w|^p\, dx$
- $\int_\Omega |\nabla w_\lambda|^p\, dx \leq c \int_\Omega |\nabla w|^p\, dx$.

Remark

Acerbi-Fusco Lemma in the power case $p \Longrightarrow$
$\|\nabla w_\lambda \chi_{\{w_\lambda \neq w\}}\|_p \leq \lambda |\{w_\lambda \neq w\}|^{\frac{1}{p}} \leq c\|w\|_{1,p} \leq K.$
So just boundedness of the above term!

Sketch of the p-harmonic approximation:

- Take h solution of the homogeneous problem in a ball B with $h = u$ on ∂B;
- Let $\gamma > 0$ s.t. $\gamma^p = \fint_B |\nabla u|^p \, dx$ and $\lambda \in [\gamma, 2^{m_0}\gamma]$ for suitable m_0. Take $w = h - u$ and w_λ s.t. $\|w_\lambda\|_\infty \leq c\lambda$ and $\fint_B \lambda^p \chi_{\{w_\lambda \neq w\}} \, dx \leq \frac{\gamma^p}{m_0}$. We consider w_λ as test function in both problems (*almost* p-harmonic estimate and p-harmonic system);
- Monotonicity of the operator, Young's inequality and useful properties of w_λ.

Let us remark the main differences with respect to the result of Duzaar and Mingione.

Remark

- We use a direct approach without a contradiction argument. This allows us to show that the constants involved in the approximation only depend on ϕ.
- We are able to preserve the boundary values of our original function. In particular, $u = h$ on ∂B.
- We show that h and u are close with respect to the gradients rather than just the functions.

Systems with critical growth (p-harmonic maps)

$$\int_\Omega \frac{\phi'(|Du|)}{|Du|} Du D\eta \, dx = \int_\Omega G\eta \, dx \tag{1}$$

for all $\eta \in C_0^\infty(\Omega)$ where $G \in L^1(\Omega)$ satisfies for a.e. $x \in \Omega$

$$|G(x)| \leq c\phi(|Du|) \tag{2}$$

Hölder regularity

Suppose $c \geq 1$ is given. Then there exists $\delta(n, N, \phi, c) > 0$ such that if $u \in W^{1,\phi}(\Omega, \mathbb{R}^N)$ satisfies the system, a Caccioppoli inequality and

$$\phi^{-1}\Big(\fint_{B_R} \phi(|\nabla u|) \, dx\Big) \leq \frac{\delta}{R}$$

on some ball $B_R \subset \Omega$, then $V(Du)$ is Hölder continuous on $B_{\frac{R}{2}}$ with exponent μ, for suitable μ depending on δ, ϕ, n, N, c.

Applications to “small solutions" (Hildebrandt, Widman, Giaquinta)

Example

If $\|u\|_\infty < c(C, \Delta_2) \Longrightarrow$ Caccioppoli inequality holds.

Principal steps

- smallness + Caccioppoli assumption $\Longrightarrow$ u is almost ϕ-harmonic;
- ϕ-harmonic approximation+ excess decay estimate for the ϕ-harmonic map $h \Longrightarrow$ Morrey-type estimate for the gradient
- convex-hull property for the functional $\Longrightarrow$ $u - h$ is continuous;
- test the system with $u - h$ and use again smallness and excess decay of h, we conclude.

Differential forms (L.Beck)

$$d^* a(x,\omega) = 0 \qquad \text{and} \qquad d\omega = 0\,, \tag{3}$$

- Ω bounded open set in $\mathbb{R}^n$;
- $\Lambda^\ell \Omega = \Lambda^\ell(T\Omega, \mathbb{R}^N)$ the vector bundle of differential ℓ-forms over the manifold Ω;
- $a\colon \Omega \times \Lambda^\ell \to \Lambda^\ell \Omega$ of class $C^0(\Lambda^\ell\Omega, \Lambda^\ell\Omega) \cap C^1(\Lambda^\ell\Omega \setminus \{0\}, \Lambda^\ell\Omega)$, satisfying some p-growth, ellipticity and continuity assumptions;
- $\omega \in L^p(\Lambda^\ell\Omega) := L^p(\Omega, \Lambda^\ell\Omega)$, $1 < p < \infty$.

Uhlenbeck's result '77

Consider

$$a(\bar{\omega}) = g(|\bar{\omega}|)\,\bar{\omega}$$

for every $\bar{\omega} \in \Lambda^{\ell}$, where the function $g \colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following continuity, ellipticity and growth conditions:

(G1) $t \mapsto g(t)$ is of class $C^0([0,\infty]) \cap C^1((0,\infty])$,

(G2) $$\nu\, t^{p-2} \le g(t) \le L\, t^{p-2}$$

and

$$\nu\, t^{p-2} \le g(t) + g'(t)\, t \le L\, t^{p-2},$$

(G3) $\exists \beta_g \in (0, \min\{1, |p-2|\})$ such that

$$|g'(s)\, s - g'(t)\, t| \le L\,(|s|^2 + |t|^2)^{\frac{p-2-\beta_g}{2}}\, |s-t|^{\beta_g}.$$

for all $s, t \in \mathbb{R}^+$, $p \ge 2$, and $0 < \nu \le L$.

Theorem ($C^{1,\alpha}$-regularity) Uhlenbeck and Hamburger '92

Given a system of Uhlenbeck structure, there exists a constant $c \geq 1$ and an exponent $\gamma \in (0,1)$ depending only on n, N, ℓ, p, L and ν such that the whenever $h \in L^p(\Lambda^\ell \Omega)$ is a weak solution of the system

$$d^*\big(g(|h|)\,h\big) = 0 \qquad \text{and} \qquad dh = 0 \qquad \text{in } \Omega\,,$$

then, for every $B_R(x_0) \subset \Omega$ and any $0 < r < R$ there holds

$$\sup_{B_{R/2}(x_0)} |h|^p \leq c \fint_{B_R(x_0)} |h|^p\,,$$

$$\Phi(h; x_0, r) \leq c \left(\frac{r}{R}\right)^{2\gamma} \Phi(h; x_0, R)\,.$$

where $\Phi(h; x_0, r)$ is the excess functional.

Principal steps

- generalization in the context of differential forms;(Gaffney's inequality,Hodge decomposition, Poincaré-type inequality, see Iwaniec-Scott-S. '99);
- generalization of the existing results concerning possibly degenerate problems (Duzaar Mingione '04 and '08);
- a unified and simplified proof of the partial regularity result for the sub- and the superquadratic case(Diening,S.,Verde).

What kind of regularity we can expect?
Partial regularity

Main Theorem

Let $\omega \in L^p(\Lambda^\ell \Omega)$, $p \in (1, \infty)$, be a weak solution

$$\Downarrow$$

ω is partially Hölder continuous .